J. Fluid Mech. (1975), *vol.* 68, *part* 4, *pp.* 695–719
Printed in Great Britain

Modal equations for cellular convection

By D. O. GOUGH,
Institute of Astronomy and Department of Applied Mathematics and Theoretical Physics, University of Cambridge

E. A. SPIEGEL
Astronomy Department, Columbia University, New York 10027

AND JURI TOOMRE
Joint Institute for Laboratory Astrophysics and Department of Astro-Geophysics, University of Colorado, Boulder

(Received 23 December 1973 and in revised form 6 September 1974)

We expand the fluctuating flow variables of Boussinesq convection in the planform functions of linear theory. Our proposal is to consider a drastic truncation of this expansion as a possibly useful approximation scheme for studying cellular convection. With just one term included, we obtain a fairly simple set of equations which reproduces some of the qualitative properties of cellular convection and whose steady-state form has already been derived by Roberts (1966). This set of 'modal equations' is analysed at slightly supercritical and at very high Rayleigh numbers. In the latter regime the Nusselt number varies with Rayleigh number just as in the mean-field approximation with one horizontal scale when the boundaries are rigid. However, the Nusselt number now depends also on the Prandtl number in a way that seems compatible with experiment. The chief difficulty with the approach is the absence of a deductive scheme for deciding which planforms should be retained in the truncated expansion.

1. Introduction

When thermal convection occurs in Nature, it is usually characterized by exceedingly large values of the Rayleigh number. For such circumstances, there are no generally accepted theoretical treatments, though mixing-length theory in one of its various forms is often used. This may be adequate when, as is often true, it is required to calculate mean quantities or to estimate transports. On the other hand, the drawbacks are evident; in particular, mixing-length theory cannot cope easily, if at all, with the various important complications that occur naturally, such as variable density, penetration and rotation. It would therefore be helpful to develop an approximation for convection which works moderately well for laboratory conditions and which, without special new assumptions, is adaptable to convection in natural circumstances. It is the aim of this paper to outline an attempt in this direction.

The approach taken here is motivated partially by observations of solar convection. There, in spite of the large Rayleigh number ($\sim 10^{20}$), one observes motion which appears cellular. Of course, the supposed cells might be the tops of thermals, or any of a variety of other possible alternatives. Nevertheless there is a certain cogency to the interpretation that cell-like motion persists in what is probably an intensely turbulent flow.

Further impetus for our approach comes from solutions of the equations of motion with the terms nonlinear in fluctuating quantities omitted: the so-called mean-field equations. The original reasons for considering these equations at all are vague and go back to some work of Malkus (see, for example, Spiegel 1962; Herring 1963), as well as to mixing-length theory itself. Nonetheless, the solutions of these equations when only one horizontal scale of motion is included give horizontal means of quantities such as temperature which are in qualitative agreement with laboratory results, even at moderately high Rayleigh numbers. And when many horizontal wavenumbers are included, it appears that the qualitative nature of heat transfer in laboratory convection is reproduced, at least for large values of the Prandtl number (Chan 1971; Spiegel 1971).

It seems reasonable to try to capitalize on the relative success of the mean-field equations by adding to them some approximation for the neglected nonlinear terms. This should at least help to remedy what appears to us to be one of their principal failures, namely that the heat transport they predict does not depend on the Prandtl number. This deficiency renders the mean-field equations unsuitable for use in stellar convection theory, for example.

Attempts to add some representation of the missing nonlinear terms to the mean-field equations have already been made. One such attempt (Spiegel 1967), though rather *ad hoc*, produced a reasonably tractable set of equations; indeed, of a form similar to that which we devise here. A more systematic approach was taken by Roberts (1966), who applied the procedure of Glansdorff & Prigogine (1964) to the convection problem with a cellular trial function. His results are just the steady-state form of the leading approximation we shall treat in some detail here.

The procedure we shall follow, though giving equations like those mentioned above, is rather simple, has been in use since Ptolemy and is sometimes called the Galerkin method. We shall simply make an expansion of the horizontal structure of the velocity and temperature fields in orthogonal functions and truncate it. Presumably, if enough terms are kept, one may regard this as a sensible approximation procedure, and later we propose to develop it in this way. Here we shall study in detail the case where only one term is retained, and we prefer to think of the approximate equations so obtained as simply defining a mathematical model of a convection cell.

In §2 we shall present the proposed expansion for Boussinesq convection and show that it is energetically consistent. From this expansion, however, we shall here retain only one term and study the resulting single-mode equations in some detail. In §3 we shall examine how motion of small but finite amplitude is described by the truncated equations and compare the results with those from the full Navier–Stokes equations. Then in §4 we shall solve the single-mode equations for large values of the Rayleigh number using matched asymptotic expansions. The results introduce a reasonable Prandtl number dependence into the heat transport predicted by the usual mean-field equations.

2. Expansion of the Boussinesq equations

We consider an incompressible fluid confined between two horizontal, rigid, perfectly conducting plates at fixed temperatures. The plate separation d is assumed to be sufficiently small that the Boussinesq approximation holds. We take d as the unit of length, d^2/κ as the unit of time, where κ is the thermometric conductivity, and ΔT, the temperature difference between the plates, as the unit of temperature. As usual, we decompose the temperature into a mean and a fluctuating part: $T = \overline{T} + \theta$, where the overbar indicates a horizontal average. The non-dimensional equations of motion then become

$$\frac{1}{\sigma}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\,.\,\nabla\mathbf{u} - \overline{\mathbf{u}\,.\,\nabla\mathbf{u}}\right) = -\nabla\varpi + R\hat{\mathbf{k}}\theta + \nabla^2\mathbf{u}, \tag{2.1}$$

$$\partial\theta/\partial t + \mathbf{u}\,.\,\nabla\theta - \overline{\mathbf{u}\,.\,\nabla\theta} = \beta w + \nabla^2\theta, \tag{2.2}$$

$$\nabla\,.\,\mathbf{u} = 0, \tag{2.3}$$

$$\frac{\partial\overline{T}}{\partial t} = -\frac{\partial\overline{w\theta}}{\partial z} + \frac{\partial^2\overline{T}}{\partial z^2}, \tag{2.4}$$

where

$$\beta = -\partial\overline{T}/\partial z, \tag{2.5}$$

and $R = g\alpha\Delta T d^3/\kappa\nu$ and $\sigma = \nu/\kappa$ are the Rayleigh and Prandtl numbers. Also g is the gravitational acceleration, α is the coefficient of thermal expansion, ν is the kinematic viscosity, $\hat{\mathbf{k}}$ is a vertical unit vector, ϖ is the deviation of the pressure from its hydrostatic value divided by the (constant) density, and (u, v, w) are the (x, y, z) components of the velocity $\mathbf{u}$ with the z axis directed along $\hat{\mathbf{k}}$. The terms describing the basic hydrostatic balance of the mean state have already been subtracted from (2.1).

In studying this system we may be interested in a variety of possible boundary conditions, but we shall restrict ourselves here to cases where the conductivities of the boundaries are far greater than that of the fluid; this permits us to regard the boundary temperatures as specified. If the upper and lower boundaries are rigid then we have

$$\left.\begin{aligned} \mathbf{u} = 0, \quad \theta = 0, \quad \overline{T} = 1 \quad \text{at} \quad z = 0,\\ \mathbf{u} = 0, \quad \theta = 0, \quad \overline{T} = 0 \quad \text{at} \quad z = 1.\end{aligned}\right\} \tag{2.6}$$

Though these are the conditions we shall use principally, we shall sometimes, for comparison with other work, apply the so-called free boundary conditions to the velocity fields, namely

$$\partial u/\partial z, \quad \partial v/\partial z, \quad w = 0 \quad \text{at} \quad z = 0, 1. \tag{2.6a}$$

Linear theory shows that instability can arise for $R \geqslant R_c = 1708$ (Chandrasekhar 1961, chap. 2) if conditions (2.6) apply. The linear equations are separable, and the horizontal variations of w and θ are described by the planforms $f_i(x, y)$, which satisfy

$$\nabla_1^2 f_i(x,y) = -a_i^2 f_i(x,y), \tag{2.7}$$

where $\nabla_1^2 \equiv \partial^2/\partial y^2 + \partial^2/\partial y^2$ and the separation constant a_i is a horizontal wave-number of the motion. Under the usual assumptions about the properties of horizontal averaging (which are based on a sort of box normalization) it can be seen that f_i and f_j are orthogonal for $a_i \neq a_j$. We may then choose the normalization

$$\overline{f_i f_j} = \delta_{ij}, \tag{2.8}$$

where δ_{ij} is a Kronecker delta. (Strictly, the f's should be labelled by the a's, which form a continuous spectrum, but we use the discrete index for convenience.) For a given a_i, there is still an infinity of solutions of (2.7), but it is possible to find a set of orthogonal functions spanning the subspace of the solutions for fixed a_i. For this subspace a second index might be desirable, but we shall not indicate it explicitly.

We now expand w and θ in terms of the f's:

$$w(x,y,z,t) = \sum_i f_i(x,y)\, W_i(z,t), \quad \theta(x,y,z,t) = \sum_i f_i(x,y)\, \Theta_i(z,t). \tag{2.9}$$

The horizontal velocity components are then expanded as

$$u(x,y,z,t) = \sum_i a_i^{-2} \frac{\partial f_i}{\partial x}\frac{\partial W_i}{\partial z}, \quad v(x,y,z,t) = \sum_i a_i^{-2} \frac{\partial f_i}{\partial y}\frac{\partial W_i}{\partial z}, \tag{2.10}$$

which ensures that the continuity equation is satisfied. From these expansions we have omitted terms which describe vertical vorticity; the corresponding expansion for ϖ is of the form of (2.9).

We may now substitute these expansions into the equations of motion and, by multiplying by appropriate f's and their derivatives and taking horizontal averages, extract the desired equations for the W_i and Θ_i. The reductions are straightforward and we shall not present them, though a reader who wishes to verify the results may find it convenient to note that, in addition to the coupling constants,

$$C_{klm} = \tfrac{1}{2}\overline{f_k f_l f_m}, \tag{2.11}$$

the quantity

$$A_{klm} = \overline{f_k\left(\frac{\partial f_l}{\partial x}\frac{\partial f_m}{\partial x} + \frac{\partial f_l}{\partial y}\frac{\partial f_m}{\partial y}\right)} \tag{2.12}$$

arises in the reductions. The A's and C's are related by

$$A_{klm} = (a_l^2 + a_m^2 - a_k^2)\, C_{klm}. \tag{2.13}$$

Having obtained the expanded equations, we eliminate the pressure by taking the double curl of the momentum equation. These are familiar reductions, and we are led to the following equations:

$$\frac{1}{\sigma}\mathscr{D}_k \frac{\partial W_k}{\partial t} + \frac{1}{\sigma}\sum_{l,m} \frac{C_{klm}}{(a_l a_m)^2}\left[a_{mkl} W_m \mathscr{D}_l \frac{\partial W_l}{\partial z} + (a_{mkl} + a_{klm})\frac{\partial W_m}{\partial z}\mathscr{D}_l W_l\right] = -Ra_k^2 \Theta_k + \mathscr{D}_k^2 W_k, \tag{2.14}$$

$$\frac{\partial \Theta_k}{\partial t} + \sum_{l,m} \frac{C_{klm}}{(a_k a_m)^2}\left[a_{klm}\Theta_l \frac{\partial W_m}{\partial z} + 2a_k^2 a_m^2 W_m \frac{\partial \theta_l}{\partial z}\right] = -\frac{\partial \overline{T}}{\partial z} W_k + \mathscr{D}_k \Theta_k \tag{2.15}$$

and
$$\frac{\partial \overline{T}}{\partial t}+\frac{\partial}{\partial z}(\sum_m W_m \Theta_m) = \frac{\partial^2 \overline{T}}{\partial z^2}, \tag{2.16}$$
where
$$\mathscr{D}_k \equiv \partial^2/\partial z^2 - a_k^2, \quad a_{klm} \equiv a_k^2(a_l^2+a_m^2-a_k^2). \tag{2.17}$$

If we now multiply (2.14) by $a_k^{-2}W_k$, integrate from $z = 0$ to $z = 1$ and sum over k, we obtain after some rearrangement
$$\frac{1}{\sigma}\frac{d}{dt}\sum_k \int_0^1 \frac{1}{2}\left[a_k^{-2}\left(\frac{\partial W_k}{\partial z}\right)^2 + W_k^2\right]dz = R\sum_k \int_0^1 \Theta_k W_k\, dz - \sum_k \int_0^1 a_k^{-2}(\mathscr{D}_k W_k)^2\, dz. \tag{2.18}$$
Likewise, multiplication of (2.15) by Θ_k followed by integration and summation yields
$$\frac{d}{dt}\sum_k \int_0^1 \tfrac{1}{2}\Theta_k^2\, dz = \sum_k \int_0^1 \beta W_k \Theta_k\, dz - \sum_k \int_0^1 \left[\left(\frac{\partial \Theta_k}{\partial z}\right)^2 + a_k^2 \Theta_k^2\right]dz. \tag{2.19}$$

These two equations are just the cellular decomposition of the usual power integrals of Boussinesq convection (Malkus 1954). Thus, the horizontal mean of the kinetic energy per unit mass is
$$\tfrac{1}{2}\overline{\mathbf{u}.\mathbf{u}} = \sum_k \frac{1}{2}\left[a_k^{-2}\left(\frac{\partial W_k}{\partial z}\right)^2 + W_k^2\right]$$
and we see that (2.18) describes the roles of buoyant work and viscous drag in the production of the kinetic energy. The power integrals hold term by term, so that energetically, at least, there is no inconsistency introduced by a truncation of the expansions. This is to be expected, since we have done little more than a Fourier analysis. The planforms are a linear combination of Fourier modes of given absolute value of the horizontal wavenumber. Hence if we wish to describe, say, a hexagon, we require only one mode, whereas six Fourier modes (three rolls) would be needed. The price paid for this advantage is that in (2.15) and (2.16) the nonlinear couplings are double sums over all modes rather than the single sums which result from a Fourier decomposition.

As stated in the introduction, we could now proceed by keeping successively more terms to see how the approximation develops. Indeed, at the time of writing we have numerical solutions for up to five modes in a variety of cases, and the results are quite complicated. But at this stage it appears that the equations for one mode may already make a useful mathematical model for convection, and we wish now to specialize to that case. Accordingly, we set all the W_k and Θ_k to zero except for one pair (W_1, Θ_1) say. Then the indices are no longer needed, and we may write
$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \mathscr{D}\right)\mathscr{D}W = -Ra^2\Theta - \frac{C}{\sigma}\left[2\frac{\partial W}{\partial z}\mathscr{D}W + W\mathscr{D}\frac{\partial W}{\partial z}\right], \tag{2.20}$$
$$\left(\frac{\partial}{\partial t} - \mathscr{D}\right)\Theta = \beta W - C\left[2W\frac{\partial \Theta}{\partial z} + \Theta\frac{\partial W}{\partial z}\right], \tag{2.21}$$
$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial z}(W\Theta) = \frac{\partial^2 \overline{T}}{\partial z^2}, \tag{2.22}$$
which are the single-mode equations to be studied here. The quantity $\mathscr{D}W$ is proportional to the horizontal vorticity. We have integrated these equations

numerically from different initial conditions for various Rayleigh numbers and the solution in each case tended to a steady state after long times. This leads us to consider here only the steady-state version

$$(D^2-a^2)^2 W = Ra^2\Theta + (C/\sigma)[2(DW)(D^2-a^2)W + W(D^2-a^2)DW], \quad (2.23)$$

$$(D^2-a^2)\Theta = -\beta W + C[2WD\Theta + \Theta DW], \quad (2.24)$$

$$\beta + W\Theta = N, \quad (2.25)$$

where $D = d/dz$ and the constant of integration N is the dimensionless total heat transport or Nusselt number. The boundary conditions (2.6) for the rigid case when combined with (2.3) yield

$$W = DW = \Theta = 0 \quad \text{at} \quad z = 0, 1, \quad (2.26)$$

while (2.6 *a*) gives

$$W = D^2W = 0 \quad \text{at} \quad z = 0, 1. \quad (2.26a)$$

We have also the integral condition

$$\int_0^1 \beta\, dz = 1. \quad (2.27)$$

In addition to these formulae, it is convenient to have available the relations

$$N = 1 + \int_0^1 W\Theta\, dz \quad (2.28)$$

and

$$\beta = 1 + \int_0^1 W\Theta\, dz - W\Theta, \quad (2.29)$$

which are obtained from (2.25) and (2.27).

We should also note that the range of possible values of C is small for the standard cells normally considered: the roll, rectangle and hexagon. Most of these planforms can be recovered from a formula of Segel & Stuart (1962):

$$f_i(x, y) = \left(\frac{2}{2+\Delta^2}\right)^{\frac{1}{2}} \left[\Delta \cos a_i y + 2 \cos \frac{3^{\frac{1}{2}} a_i x}{2} \cos \frac{a_i y}{2}\right], \quad (2.30)$$

where $\Delta \to \infty$ leads to the roll, $\Delta = 0$ gives a particular rectangle and $\Delta = 1$ a hexagon. We have for these forms

$$C = \tfrac{3}{4}\Delta[2/(2+\Delta^2)]^{\frac{3}{2}}, \quad (2.31)$$

which vanishes at $\Delta = 0, \infty$ and reaches a maximum value of $6^{-\frac{1}{2}}$ for $\Delta = 1$. We shall regard these values of C as representative, though it is also possible to regard C as a measure of 'eddy effects' and treat it as a parameter.

For $C = 0$, the system (2.23)–(2.25) reduces to what are sometimes called the 'single-α mean-field equations', which have been discussed in detail (Herring 1963, 1964, 1966; Howard 1965; Roberts 1966; Stewartson 1966; Elder 1969; Van der Borght 1971; Murphy 1971 *a*). Moreover, the system has been derived by Roberts (1966) using a procedure proposed by Glansdorff & Prigogine (1964). For $C \neq 0$, Roberts has given some numerical solutions at modest R, and Murphy (1971 *b*) has gone to somewhat higher R for free–free conditions using a truncated sine series in z.

3. Solutions at small amplitude

Rayleigh (1916) and others have given solutions of the Boussinesq equations for motion of infinitesimal amplitude. When R exceeds a certain critical value R_c, linear theory predicts exponential growth for planforms in a particular band of horizontal wavenumbers (Chandrasekhar 1961, chap. 2). In the N, R plane, steady finite amplitude solutions bifurcate from the branch of static solutions at $N = 1$, $R = R_c$. At the point of bifurcation the steady convective solutions have zero amplitude and for R near R_c they should have small amplitude. Malkus & Veronis (1958) used this small amplitude as an expansion parameter in perturbation theory to find nonlinear solutions in the neighbourhood of R_c. Their work was restricted to the case of stress-free boundaries. The extension to the rigid-boundary case, for which (2.26) apply, was made by Schlüter, Lortz & Busse (1965). In this section we examine the analogous expansions of (2.23)–(2.25). We shall outline the calculation for the rigid boundary conditions (2.26), and then quote for comparison the Nusselt number found when the free boundary conditions are used instead. The point of this exercise is to see how well the solutions of the single-mode equations, which are derived from the most extreme simplification of the horizontal structure of the flow, approximate solutions of the full Boussinesq equations at low R.

We assume that the velocity and the temperature fluctuation have an amplitude $\mathscr{A}$ and write

$$W = \mathscr{A}\hat{W}, \quad \Theta = \mathscr{A}\hat{\Theta}. \tag{3.1}$$

To be specific we might consider $\mathscr{A}$ to be the norm of the velocity. Then (2.23)–(2.25) become

$$(D^2-a^2)^2\hat{W} - Ra^2\hat{\Theta} = (\mathscr{A}C/\sigma)[2(D\hat{W})(D^2-a^2)\hat{W} + \hat{W}(D^2-a^2)D\hat{W}], \tag{3.2}$$

$$(D^2-a^2)\hat{\Theta} + \beta W = \mathscr{A}C[2\hat{W}D\hat{\Theta} + \hat{\Theta}D\hat{W}], \tag{3.3}$$

$$\beta - N = -\mathscr{A}^2\hat{W}\hat{\Theta}. \tag{3.4}$$

Likewise (2.28) and (2.29) are written as

$$N = 1 + \mathscr{A}^2\int_0^1 \hat{W}\hat{\Theta}\,dz, \tag{3.5}$$

$$\beta = 1 + \mathscr{A}^2\left(\int_0^1 \hat{W}\hat{\Theta}\,dz - \hat{W}\hat{\Theta}\right) \tag{3.6}$$

and the boundary conditions (2.26) apply to $\hat{W}$ and $\hat{\Theta}$. It is convenient to combine (3.2), (3.3) and (3.6) into the single equation

$$(D^2-a^2)^3\hat{W} + Ra^2\hat{W} = (\mathscr{A}C/\sigma)(D^2-a^2)[2(D\hat{W})(D^2-a^2)\hat{W} + \hat{W}(D^2-a^2)D\hat{W}]$$

$$+\mathscr{A}Ra^2C(2\hat{W}D\hat{\Theta} + \hat{\Theta}D\hat{W}) - \mathscr{A}^2Ra^2\left(\int_0^1 \hat{W}\hat{\Theta}\,dz - \hat{W}\hat{\Theta}\right)\hat{W}. \tag{3.7}$$

We now expand $\hat{W}$, $\hat{\Theta}$, β, N and R in powers of $\mathscr{A}$, for example

$$\hat{W} = \hat{W}_0 + \mathscr{A}\hat{W}_1 + \mathscr{A}^2\hat{W}_2 + \dots,$$

substitute these expansions into the equations and the boundary conditions, and equate separately to zero the coefficients of each power of $\mathscr{A}$.

We see immediately that the boundary conditions

$$\hat{W}_n, \quad D\hat{W}_n, \quad \hat{\Theta}_n = 0 \quad \text{at} \quad z = 0, 1 \tag{3.8}$$

apply at each order. The leading terms of (3.5) and (3.6) give $\beta_0 = 1$ and $N_0 = 1$, while the leading order of (3.7) is

$$L\hat{W}_0 \equiv [(D^2 - a^2)^3 + R_0 a^2]\hat{W}_0 = 0, \tag{3.9}$$

which is the equation for marginal stability and yields, in conjunction with the boundary conditions, a set of eigenvalues for R_0, the lowest of which, R_c, occurs at a specific value a_c of a. Once $\hat{W}_0$ is known, $\hat{\Theta}_0$ can be obtained, and $\hat{W}_0$ and $\hat{\Theta}_0$ may be written in the form

$$\left.\begin{aligned} \hat{W}_0 &= \sum_{r=1}^{3} \hat{A}_r \cosh q_r(z - \tfrac{1}{2}), \\ \hat{\Theta}_0 &= \sum_{r=1}^{3} \hat{B}_r \cosh q_r(z - \tfrac{1}{2}); \end{aligned}\right\} \tag{3.10}$$

A_r, B_r and q_r are complex constants which depend on the boundary conditions used and are given, for example, by Reid & Harris (1958) and Chandrasekhar (1961, § 15) for the gravest mode, whose velocity and temperature fluctuations vanish nowhere other than at $z = 0$ and 1. It may be shown that $\hat{\Theta}_0$ is adjoint to $\hat{W}_0$ with respect to L and the boundary conditions (3.8), and that for given R_0 and a the solution, apart from a constant amplitude factor, is unique.

The higher-order equations derived from (3.7) and (3.2) may be written as

$$L\hat{W}_n = -R_n a^2 \hat{W}_0 + P_n, \tag{3.11}$$

$$\hat{\Theta}_n = (R_0 a^2)^{-1} (D^2 - a^2)^2 \hat{W}_n + Q_n, \tag{3.12}$$

where P_n and Q_n depend on expansion coefficients of $\hat{W}$, $\hat{\Theta}$ and R of all orders up to $n-1$. The necessary and sufficient condition for the existence of a solution of (3.11) subject to (3.8) is that the right-hand side be orthogonal to the adjoint function $\hat{\Theta}_0$. Thus

$$R_n a^2 \int_0^1 \hat{W}_0 \hat{\Theta}_0 dz = \int_0^1 P_n \hat{\Theta}_0 dz, \tag{3.13}$$

which determines R_n. It can be demonstrated that the integral on the left side of this equation cannot be zero. Also it follows from the structure of P_n and the zero-order solution (3.10) that P_1 is of the form

$$P_1 = \sum_{r,s=1}^{3} [P_{rs}^{+} \sinh (q_r + q_s)(z - \tfrac{1}{2}) + P_{rs}^{-} \sinh (q_r - q_s)(z - \tfrac{1}{2})],$$

where the $P_{rs}^{\pm}$ are constants. This is an odd function of $z - \frac{1}{2}$, so $R_1 = 0$. In view of this we can immediately write down a particular integral of (3.11) which is a sum of terms proportional to $\sinh(q_r \pm q_s)(z-\frac{1}{2})$. The complementary function and the part of $\hat{\Theta}_1$ obtained from (3.12) arising directly from the complementary function are sums of terms proportional to $\cosh q_r(z-\frac{1}{2})$ whose coefficients could be determined by applying the boundary conditions. Thus each term in P_2 involving the complementary function is an odd function of $z-\frac{1}{2}$ and is orthogonal to $\hat{\Theta}_0$; hence to obtain R_2 it is not necessary to evaluate the complementary function. The substitution of the particular integral into (3.13) is quite straightforward and will not be reproduced here.

To compute the correction to the Nusselt number we expand (3.5), and obtain

$$N_1 = 0, \quad N_2 = \int_0^1 |\hat{W}_0 \hat{\Theta}_0 dz.$$

We have thus found the deviations of N and R from their marginal values implied by a steady motion of amplitude $\mathscr{A}$:

$$R = R_0 + \mathscr{A}^2 R_2, \quad N = 1 + \mathscr{A}^2 N_2.$$

From these we can eliminate $\mathscr{A}$ to obtain

$$N - 1 = N_2 \frac{R_0}{R_2}\left(\frac{R}{R_0} - 1\right). \tag{3.14}$$

After some elementary reductions, this relation can be written in the form

$$N - 1 = [\alpha_1 + C^2(\alpha_2 + \alpha_3 \sigma^{-1} + \alpha_4 \sigma^{-2})]^{-1} (R/R_0 - 1), \tag{3.15}$$

where the α's are numerical factors depending on a but not on C and σ. For $a = a_c = 3{\cdot}117$,

$$\alpha_1 = 6{\cdot}92 \times 10^{-1}, \quad \alpha_2 = 9{\cdot}32 \times 10^{-1}, \quad \alpha_3 = 8{\cdot}10 \times 10^{-2}$$

and

$$\alpha_4 = 4{\cdot}01 \times 10^{-1}.$$

For rolls ($C = 0$) and hexagons ($C = 6^{-\frac{1}{2}}$) we find, respectively,

$$N - 1 = (0{\cdot}692)^{-1} (R/R_c - 1), \tag{3.15a}$$

$$N - 1 = (0{\cdot}847 + 0{\cdot}135\sigma^{-1} + 0{\cdot}0668\sigma^{-2})^{-1} (R/R_c - 1). \tag{3.15b}$$

From the full Boussinesq equations, Schlüter *et al.* (1965) obtained for these planforms

$$N - 1 = (0{\cdot}699 - 0{\cdot}00476\sigma^{-1} + 0{\cdot}0083\sigma^{-2})^{-1} (R/R_c - 1), \tag{3.16a}$$

$$N - 1 = (0{\cdot}894 + 0{\cdot}0496\sigma^{-1} + 0{\cdot}0679\sigma^{-2})^{-1} (R/R_c - 1). \tag{3.16b}$$

The most striking aspect of this comparison is the failure of the modal equations to produce a Prandtl number dependence for $C = 0$. For hexagons, the solution of the modal equations is adequate for many purposes, and as (3.15) is somewhat easier to obtain than (3.16), the simplified system is likely to be useful for more complicated problems.

If both the boundaries are free the boundary conditions in (2.26) are replaced by (2.26 a) and we find by a similar but simpler calculation

$$N = 1+\frac{2(R/R_0-1)}{1+\{[1+q_2^2/(\sigma q_1^2)]^2 q_1^6/(q_2^6-q_1^6)+1\}}, \tag{3.17}$$

where
$$R_0 = (a^2+\pi^2)^3/a^2, \quad q_n^2 = a^2+n^2\pi^2. \tag{3.18), (3.19}$$

For rolls and hexagons Malkus & Veronis (1958) give explicit results for $\sigma = \infty$, in which case the agreement is excellent. However, in the case of rectangles, $C = 0$ and the present procedure fails to give any dependence of N on σ, in contrast to the results of Malkus & Veronis.

4. Solutions for large R

The case of most interest in studying natural convection is that of very large R, and we shall examine in this section the predictions of (2.23)–(2.25) for $R\to\infty$. For the case $C = 0$, asymptotic solutions for $R\to\infty$ have already been found. For rigid boundaries Roberts (1966) and Stewartson (1966) found that N varies as $(Ra^2 \ln Ra^2)^{\frac{1}{5}}$ for large R and for suitably restricted wavenumbers a, while Howard (1965) found a variation like $R^{\frac{1}{3}}$ for free boundaries. In both cases N is of course independent of σ. For $C \neq 0$, the inertial terms have some effect and naturally the asymptotic development is more involved.

The representation of the fluctuation advection terms introduces a vertical asymmetry into the solutions, which is already apparent in the finite amplitude results of the previous section. However the steady equations (2.23)–(2.25) are invariant under the transformation $z\to 1-z$, $W\to -W$, $\Theta\to-\Theta$, $\overline{T}\to 1-\overline{T}$ (Roberts 1966). These equations are also invariant under the transformation $C\to -C$, $z\to 1-z$, $W\to W$, $\Theta\to\Theta$, $\overline{T}\to 1-\overline{T}$, but this does not lead to another solution. As in § 3 we only seek solutions for which W and Θ do not vanish other than at $z = 0, 1$ and consider the solution with W, Θ and C positive. When we refer later to aspects of the solution near a particular boundary it should be realized that there exists a corresponding solution with the designations 'upper' and 'lower' interchanged.

The scalings

To treat the problem of large R we shall use matched asymptotic expansions, for which purpose we introduce

$$\epsilon = (Ra^2N\sigma/C)^{-\frac{1}{3}}, \quad \Psi = \epsilon W, \quad F = (\epsilon N)^{-1}\Theta, \quad B = N^{-1}\beta \tag{4.1}$$

into (2.23)–(2.25) and multiply (2.23) and (2.24) respectively by Ψ and F to obtain

$$\epsilon\Psi(D^2-a^2)^2\Psi = (C/\sigma)\{\Psi F + D[\Psi^2(D^2-a^2)\Psi]\}, \tag{4.2}$$

$$\epsilon^2 F(D^2-a^2)F = -BF\Psi + \epsilon C D(F^2\Psi), \tag{4.3}$$

$$B+\Psi F = 1. \tag{4.4}$$

The rigid boundary conditions (2.26) become

$$\Psi = D\Psi = F = 0 \quad \text{at} \quad z = 0, 1 \tag{4.5}$$

and we have from (2.27)

$$\frac{1}{N} = \int_0^1 B\,dz. \tag{4.6}$$

We now study the nature of the solutions as $\epsilon \to 0$. Before proceeding to the outer expansions, we shall anticipate here the scaling needed for the boundary layers. Near the lower boundary we let

$$\left.\begin{aligned} z = \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}\zeta, \quad \Psi = \epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}\psi, \quad F = \epsilon^{-\frac{1}{2}}\lambda^{-\frac{1}{6}}f, \\ B = b, \quad \lambda = \tfrac{3}{2}\ln\epsilon^{-1}. \end{aligned}\right\} \tag{4.7}$$

The boundary-layer equations are then

$$\psi\psi^{\text{iv}} - \frac{C}{\sigma}(\psi^2\psi'')' = \lambda^{-1}\frac{C}{\sigma}\psi f, \tag{4.8}$$

$$ff'' + bf\psi - C(f^2\psi)' = 0, \tag{4.9}$$

$$b + \psi f - 1 = 0, \tag{4.10}$$

where a prime denotes differentiation with respect to the argument (here ζ) and we have neglected terms which are $O(a^2\epsilon\lambda^{-\frac{1}{3}})$.

Near the upper boundary we set

$$1 - z = \epsilon^{\frac{2}{3}}\eta, \quad \Psi = \epsilon^{\frac{2}{3}}\phi, \quad F = \epsilon^{-\frac{2}{3}}g, \quad B = b, \tag{4.11}$$

so that the equations for the upper boundary layer are

$$\phi\phi^{\text{iv}} + \frac{C}{\sigma}(\phi^2\phi'')' = \epsilon^{\frac{2}{3}}\frac{C}{\sigma}\phi g, \tag{4.12}$$

$$gg'' + bg\phi + C(g^2\phi)' = 0, \tag{4.13}$$

$$b + \phi g - 1 = 0. \tag{4.14}$$

The interior solution

We need first to find the appropriate asymptotic sequence for the solution away from the boundaries. The leading term Ψ_0 can be inferred from (4.1)–(4.3) by letting $\epsilon \to 0$. From (4.3) we find $B_0\Psi_0 F_0 = 0$, which suggests the choice $B_0 = 0$. Then (4.4) implies that $\Psi_0 F_0 = 1$. We can then neglect the left-hand side of (4.2) and integrate once to obtain

$$\Psi_0^2(D^2 - a^2)\Psi_0 = z_0 - z, \tag{4.15}$$

where z_0 is an integration constant. An equation of the same form can be obtained by integrating (4.2) from 0 to z and making the identification

$$\epsilon\frac{\sigma}{C}\int_0^z \Psi(D^2 - a^2)^2\Psi\,dz + \frac{1 - \overline{T}}{N} = z_0. \tag{4.16}$$

Since B_0 is zero, $\overline{T}$ is constant to leading order in the interior; the integral in (4.16) is also sensibly constant in the interior since its main contribution comes from the boundary layer. Hence, expression (4.16) gives the constant z_0. Now we expect, and shall verify below, that N^{-1} is proportional to the boundary-layer thickness $\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}$, while on introducing the scaled variables (4.7) we find that the integral in (4.16) is $O(\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}})$. Thus $z_0 = O(\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}})$, and we can neglect it in (4.15). However, these estimates give us some indication of the kind of higher-order terms to be expected in the development of Ψ.

The details of this asymptotic development for Ψ are given in appendix A. To illustrate the calculation we simply consider here the leading term Ψ_0, which is obtained from (4.15) by neglecting z_0. This is adequate for obtaining the leading term in the dependence of N on R. Near $z = 0$, (4.15) admits solutions which vanish for $z = 0$ which are of the form

$$\Psi_0 = z\left(3\ln\frac{K_1}{z} - 2\ln\ln\frac{K_1}{z} + \ldots\right)^{\frac{1}{3}} + \ldots, \tag{4.17}$$

while near $z = 1$, the solution which vanishes at $z = 1$ is

$$\Psi_0 = (\tfrac{9}{2})^{\frac{1}{3}}(1-z)^{\frac{2}{3}}[1 + K_2(1-z)^{\frac{2}{3}} + \tfrac{1}{3}(1-z) + \ldots] + \ldots, \tag{4.18}$$

where K_1 and K_2 are constants. Though (4.17) and (4.18) give the behaviour at the edges of the interior, we have no guarantee that the two limiting behaviours correspond to one solution and we need reassurance on this point. For $a \gg 1$ we can apply matched asymptotic expansions to (4.15) and see that indeed all is well. This procedure also determines K_1 and K_2. However, as we have no explicit need here for these results, we defer that calculation to appendix B.

An interesting feature of this approximation for the interior solution is that it predicts a bump in the mean temperature profile near one boundary (here the lower one). To see this we recall that $F_0\Psi_0 = 1$ and find from (4.3) that

$$B = \epsilon CDF_0 + \ldots. \tag{4.19}$$

Since $DF_0 = D(\Psi_0^{-1})$ we see from (4.17) that near $z = 0$

$$B \simeq -\epsilon Cz^{-2}[3\ln(K_1/z) + \ldots]^{-\frac{1}{3}} \tag{4.20}$$

while near $z = 1$, $B > 0$. This is just the interior part; the actual value of B must be brought to unity at both boundaries with a boundary-layer solution. Thus $\overline{T}$ must have a bump near the lower boundary, but does not have one near the upper boundary. (The reverse is true for $C < 0$.)

We now express the interior solution in the boundary-layer variables defined by (4.7) and (4.11). From (4.17) we find that, near $z = 0$, $\Psi_0 = \epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}\zeta + O(\ln\lambda/\lambda)$. In fact the higher-order terms in the outer asymptotic sequence contribute terms to this leading order. The calculation in appendix A gives us

$$\Psi = \epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}(\zeta - A_1\ln\zeta - \zeta_0 + \ldots) + O(\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}) \tag{4.21}$$

near $\zeta = 0$, whence

$$F = \epsilon^{-\frac{1}{2}}\lambda^{-\frac{1}{6}}\zeta^{-1}\left(1 + \frac{A_1\ln\zeta}{\zeta} + \frac{\zeta_0}{\zeta} + \ldots\right) + O(\epsilon^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}), \tag{4.22}$$

where A_1 and ζ_0 are constants. Likewise, near $z = 1$, the interior solution expressed in the appropriate boundary-layer variables is

$$\Psi = \epsilon^{\frac{2}{5}}(\tfrac{9}{2})^{\frac{1}{3}}\eta^{\frac{2}{3}}[1 + (\tfrac{2}{9})^{\frac{1}{3}}K_3\eta^{-1} - (\tfrac{2}{9})^{\frac{1}{3}}(4\sigma/15C)\eta^{-\frac{5}{3}} + \ldots] + O(\epsilon^{\frac{4}{5}}), \tag{4.23}$$

$$F = \epsilon^{-\frac{2}{5}}(\tfrac{2}{9})^{\frac{1}{3}}\eta^{-\frac{2}{3}}[1 - (\tfrac{2}{9})^{\frac{1}{3}}K_3\eta^{-1} + \tfrac{2}{3}(\tfrac{2}{9})^{\frac{1}{3}}(2\sigma/5C - C)\eta^{-\frac{5}{3}} + \ldots] + O(1), \tag{4.24}$$

where K_3 is an arbitrary constant.

Boundary-layer solutions

Near the bottom boundary layer we introduce the scalings (4.7) and obtain (4.8)–(4.10). We consider here only the matching of the leading terms, for which purpose (4.8) may be approximated as

$$\psi\psi^{\text{iv}} - (C/\sigma)(\psi^2\psi'')' = 0. \tag{4.25}$$

This is to be solved subject to the conditions

$$\psi(0) = \psi'(0) = 0, \quad \psi \sim \zeta - A_1 \ln \zeta - \zeta_0 \quad \text{as} \quad \zeta \to \infty. \tag{4.26}$$

Integration yields

$$\psi''' - (C/\sigma)[\psi\psi'' + \tfrac{1}{2}(\psi'^2 - 1)] = 0. \tag{4.27}$$

With the substitutions

$$\psi(\zeta) = (\sigma/C)^{\frac{1}{2}}\chi(\xi), \quad \xi = (C/\sigma)^{\frac{1}{2}}\zeta, \tag{4.28}$$

we obtain

$$\frac{d^3\chi}{d\xi^3} - \chi\frac{d^2\chi}{d\xi^2} - \frac{1}{2}\left[\left(\frac{d\chi}{d\xi}\right)^2 - 1\right] = 0, \tag{4.29}$$

which is free of parameters. The boundary conditions are now

$$\chi(0) = [d\chi/d\xi]_{\xi=0} = 0, \quad \chi \sim \xi - A\ln\xi - \xi_0 \quad \text{as} \quad \xi \to \infty, \tag{4.30}$$

where

$$A = (C/\sigma)^{\frac{1}{2}}A_1, \quad \xi_0 = [\tfrac{1}{2}A_1 \ln(\sigma/C) + \zeta_0](C/\sigma)^{\frac{1}{2}} \tag{4.31}$$

are as yet unknown.

Equation (4.40) is of the Falkner–Skan type and is best solved numerically. We have carried out two independent numerical solutions in collaboration with Dr K. Grossman and they yield $A = 0{\cdot}335$, $\xi_0 = 1{\cdot}19$ and $d^2\chi/d\xi^2 = 0{\cdot}729$ at $\xi = 0$. The function χ is exhibited in figure 1.

The solution in the lower boundary layer is completed by solving (4.9), which can be rewritten as

$$f'' - 2C\psi f' - (\psi^2 + C\psi')f = -\psi, \tag{4.32}$$

subject to

$$f(0) = 0, \quad f \sim \zeta^{-1} \quad \text{as} \quad \zeta \to \infty. \tag{4.33}$$

For purposes of numerical solution it is convenient to introduce

$$\tilde{f}((C/\sigma)^{\frac{1}{2}}\zeta) = \tilde{f}(\xi) = (\sigma/C)^{\frac{1}{2}}f(\zeta). \tag{4.34}$$

Then

$$\tilde{f}'' = (-\sigma^2/C^2)(1 - \chi\tilde{f})\chi + \sigma(2\chi\tilde{f}' + \chi'\tilde{f}), \tag{4.35}$$

where a prime indicates differentiation with respect to the argument (here ξ). Since we have χ, a function of ξ alone, it is straightforward to solve (4.46) numerically for $\tilde{f}$ for various values σ and C. Some sample solutions for $\tilde{f}$ are shown in figure 1, for $C = 6^{-\frac{1}{2}} = 0{\cdot}408$ and $\sigma = 10^{-2}$, 1, 10^2 and 10^4.

The matching to the top boundary layer proceeds in a similar fashion and leads to $K_3 = -0{\cdot}927(\sigma/C)^{\frac{3}{5}}$. We need not repeat the discussion, but we should mention that the numerical solutions in both cases (upper and lower boundary) require some care.

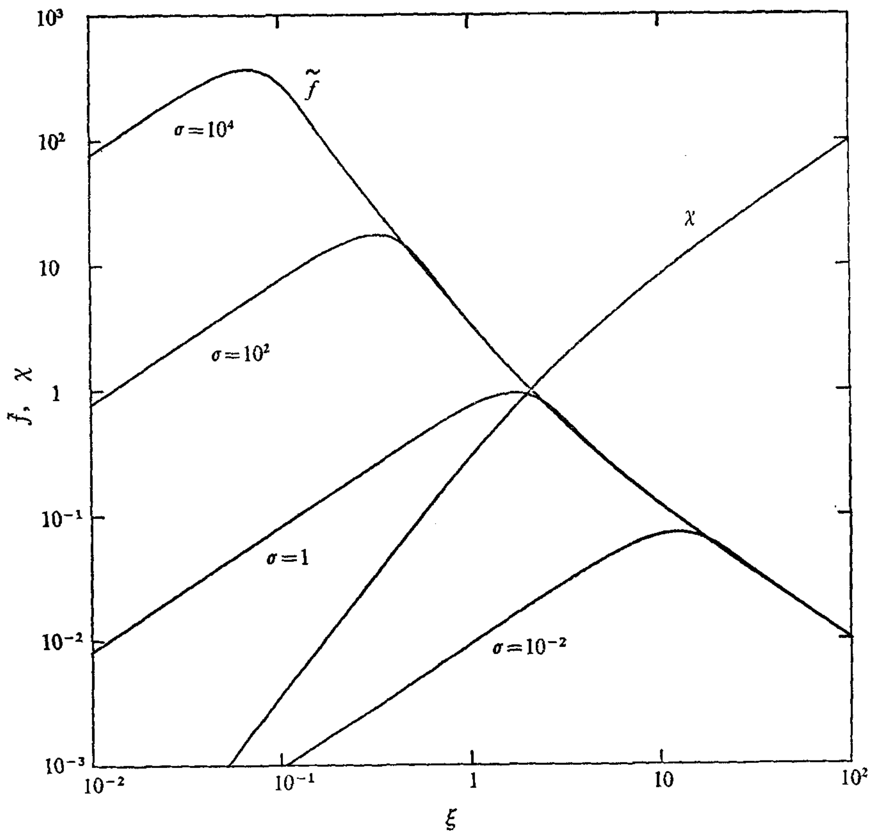

FIGURE 1. The lower boundary-layer functions $\tilde{f}$ and χ from (4.35) and (4.29) are displayed over a representative interval of the independent variable ξ. The functions $\tilde{f}$ are for $C = 0{\cdot}408$ (hexagonal planform), and for the Prandtl numbers $\sigma = 10^{-2}, 1, 10^2$ and 10^4. The function χ in these numerical solutions is independent of C and σ, and has the limiting form $\chi \propto \xi^2$ as $\xi \to 0$.

The Nusselt number

We may evaluate N from (4.6) by separating the integral into the contributions from the interior and the two boundary layers:

$$\frac{1}{N} = \int_0^1 B\,dz = \int_0^1 (1-\Psi F)\,dz \sim \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}\int_0^\infty (1-\psi f)\,d\zeta + O(\epsilon) + \epsilon^{\frac{2}{3}}\int_0^\infty (1-\phi g)\,d\eta$$

$$\sim \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}(\sigma/C)^{\frac{1}{2}}\int_0^\infty (1-\chi\tilde{f})\,d\xi, \qquad (4.36)$$

the term $O(\epsilon)$ coming from the interior [see (4.19)]. The quantity

$$k = (\sigma/C)^{\frac{1}{2}}\int_0^\infty (1-\chi\tilde{f})\,d\xi \qquad (4.37)$$

can be evaluated, for different σ and C, from the boundary-layer solutions. In table 1 we give values of k for a large range of σ and for a few values of C. From

$\sigma \backslash C$	0·408	0·300	0·200	0·100
10^5	12·64	13·15	13·96	15·58
10^4	8·63	8·97	9·52	10·63
10^3	5·90	6·14	6·51	7·26
10^2	4·05	4·21	4·46	4·97
10	2·82	2·92	3·09	3·43
1	2·01	2·07	2·18	2·40
10^{-1}	1·53	1·56	1·61	1·74,
10^{-2}	1·28	1·28	1·30	1·37
10^{-3}	1·17	1·16	1·16	1·19
10^{-4}	1·13	1·11	1·11	1·11
10^{-5}	1·11	1·10	1·09	1·08

TABLE 1. Dependence of k, the integral expression (4.37), upon C and σ. The values of k may be used in evaluating the Nusselt number N in (4.38). The values of k for a free boundary, given by (4.41), at the above C values are 1·10, 1·09, 1·07 and 1·06

the analysis we know that k depends on C and σ/C but not on σ alone. We see from table 1 that, for large σ/C, k varies approximately like $(\sigma/C)^{\frac{1}{6}}$, with a coefficient that varies weakly with C and is about 1·60 at $C = 0{\cdot}408$. For very small σ, k is insensitive to either σ or C.

We may now refer to definition (4.1) to establish that

$$N \sim k^{-1}(\tfrac{3}{5}\mathscr{R}\ln\mathscr{R})^{\frac{1}{5}} \quad \text{as} \quad \mathscr{R}\to\infty, \tag{4.38}$$

where

$$\mathscr{R} = Ra^2\sigma/kC.$$

If the leading term from the upper boundary layer is taken into account, the factor k in (4.38) and in $\mathscr{R}$ must be replaced by

$$k+\mathscr{R}^{-\frac{1}{25}}(\ln\mathscr{R})^{\frac{4}{25}}k'+\dots,$$

where

$$k' = \int_0^\infty (1-\phi g)\,d\eta,$$

while if we include the next term from the lower boundary layer the whole formula must be multiplied by $1-\frac{7}{15}(\ln\ln\mathscr{R}/\ln\mathscr{R})+\dots$, at least when C, σ and a are of order unity.

Since B is the gradient of $\overline{T}$, we see from (4.36) that as $R\to\infty$ all of the drop in $\overline{T}$ occurs in the lower boundary layer. Thus at sufficiently large R the bump in $\overline{T}$ implied by (4.19) causes $\overline{T}$ to lie slightly outside the range defined by the boundary conditions. Also, to leading order the Nusselt number is independent of whether the upper boundary is rigid or free.

Other parameter ranges

In obtaining (4.38) we have treated a, C and σ/C as $O(1)$, though these restrictions can be relaxed somewhat without loss of qualitative information. For example, when $\sigma \ll 1$ (C being fixed and non-zero) $k \simeq 1{\cdot}1$, which shows that to within the present accuracy N depends on $R\sigma$ but not on R or σ separately (a being fixed). However, the explicit dependence of N on $R\sigma$ indicated by (4.38) holds only for $R\sigma \gg 1$.

When $\sigma R/C$ is small a totally different scaling of (2.23)–(2.25) is necessary because now most of the heat is transported by conduction. There are no thermal boundary layers and although W depends on which of the sets of boundary conditions (2.26) or (2.26a) is imposed, Θ and N do not. In particular,

$$N \sim 1+\Lambda(\sigma R/C)^2, \tag{4.39}$$

where Λ is a function of a alone, when R is large. It was found by numerical integration of the interior equations that Λ has a single maximum $\Lambda_m = 2{\cdot}496 \times 10^{-6}$, which occurs at $a = 2{\cdot}370$.

For large σ/C it is not hard to show that there is virtually no change in the results and that the form of (4.38) remains valid as long as $\sigma/C < (Ra^2)^{\frac{3}{5}}$. When σ becomes very large compared with C, N becomes essentially independent of σ; this is true also when $C \to 0$. In fact, as C tends to zero, the form of the dependence of N on Ra^2 is just that found by Stewartson (1966) and Roberts (1966). However, the asymptotic analysis presented here is not valid when $C \ll \sigma(Ra^2)^{-\frac{3}{5}}$, since in that limit a different balance of terms holds. Viscous forces balance buoyancy both in the interior and in the boundary layers, and the matching is quite different. Nevertheless, the Rayleigh number dependence when $C=0$ is the same as expression (4.38), though the coefficient k is different from that obtained simply by letting $C \to 0$ in (4.37). There are also differences in logarithmic terms, but if anything, the similarities rather than the differences between the results so differently obtained are remarkable.

The dependence of N on a implied by (4.38) is also of interest and implies a general increase in N with a in the asymptotic range considered here. This clearly must break down at large enough a since we know that, for large R, $N = 1$ when $a \gtrsim R^{\frac{1}{4}}$. Moreover, the small amplitude studies are applicable sufficiently near to the marginally stable solutions, even at large R, and we find no subcritical solutions. Thus, (4.38) must break down before a increases to $R^{\frac{1}{4}}$ – but where?

From our discussion of the interior equation for large a (appendix B) we see that boundary layers of thickness a^{-1} arise. When $a = O(R^{\frac{1}{4}})$, (4.1) and (4.38) imply that $\epsilon^{\frac{5}{12}}(\ln \epsilon^{-1})^{\frac{1}{36}} = O(a^{-1})$, which is larger than the thicknesses of the boundary layers already considered. Hence no significant change in the structure of the viscous boundary layer results, though $\ln K_1$ and K_2 in (4.17) and (4.18) are no longer of order unity (cf. appendix B); the order of the asymptotic sequence derived in appendix A and the subsequent matching are modified. Furthermore, when $a \geqslant O[R^{\frac{3}{14}}(\ln R)^{\frac{1}{28}}]$ equation (4.15) is valid only in an intermediate boundary layer of thickness $a^{-\frac{14}{5}} R^{\frac{3}{5}} (\ln R)^{\frac{1}{10}}$, which is always greater than a^{-1}. In the true interior the viscous and buoyancy forces balance and Ψ is asymptotically constant. The temperature is no longer asymptotically constant, and decreases linearly with z. Consequently, the interior contributes in leading order to the integral for N^{-1} in (4.36), as it does when $C = 0$ (cf. Stewartson 1966), the amount it contributes being $a^6\epsilon^3\sigma/C$. The intermediate boundary layers do not contribute in leading order and we find that formula (4.38) for N should be multiplied by $(1-a^4/R)[(1-3\nu)/(1+2\nu)]^{\frac{1}{5}}$ and $\mathscr{R}$ should be replaced by $(1-a^4/R)\mathscr{R}$, where $\nu = (\ln a/\ln R)$. This gives $a_{\max} \simeq (\frac{1}{13}R)^{\frac{1}{4}}$. The additional adjustments that are made when higher-order terms in the expansion are taken into account still apply provided that $a \ll R^{\frac{4}{17}}(\ln R)^{\frac{2}{51}}$, except that the factor $\frac{7}{15}$ must be multiplied by $(1+2\nu)/(1-3\nu)$; for larger a the intermediate boundary layers near the lower boundary also contribute to the correction to k.

Free boundaries

In many theoretical studies of convection it is assumed for mathematical convenience that at the boundaries there are no tangential stresses, and this is commonly referred to as the free boundary condition, even though the boundaries are not permitted to deform. For 'free' boundaries the dynamic boundary conditions are (2.26 a). When $C = 0$, these conditions result in a qualitatively different dependence of N on R than do the rigid conditions. Howard (1965) finds that for free boundaries $N \sim R^{\frac{1}{3}}$ when $R \to \infty$ and $C = 0$, and that, at fixed R, N is maximum for $a = O(1)$. [For $C = 0$, N is maximum at $a = O(R^{\frac{1}{4}})$ with rigid boundaries (Roberts 1966; Stewartson 1966).] For purposes of comparison with these studies we sketch here asymptotic results for free boundaries when $C \neq 0$.

The ordering of the terms in the body of the fluid (i.e., the asymptotic sequence) is the same for the free boundary case as for the rigid. Thus the analysis is the same as above up to the point where (4.27) must be solved. But now the conditions on ψ are

$$\psi(0) = \psi''(0) = 0, \quad \psi \sim \zeta - A_1 \ln \zeta - \zeta_0 \quad \text{as} \quad \zeta \to \infty. \tag{4.40}$$

This time the solution is $\psi = \zeta$ with $A_1 = 0$ and $\zeta_0 = 0$. The completion of this analysis is outlined in appendix C, where it is found that (4.38) holds but now

$$k = 2^{-\frac{1}{2}}[\Gamma(\tfrac{3}{4})]^2(1+C^2)^{\frac{1}{4}} = 1{\cdot}062(1+C^2)^{\frac{1}{4}}, \tag{4.41}$$

which is independent of σ. This means that, when $\sigma = O(1)$, N depends on σ and R only through the combination $R\sigma$ as in the case with rigid boundaries at low σ. This behaviour does not persist for very large σ, and the analysis shows that modifications are expected when σ is of order $C(Ra^2)^{\frac{2}{3}}$. The general treatment of the problem is difficult and we have simply examined the limit $\sigma \gg C(Ra^2)^{\frac{2}{3}}$.

In this limit the inertial term in the momentum equation is negligible and the interior equation for Ψ is the same as when $C = 0$. On the other hand, the boundary-layer equations are the same as for $C \neq 0$ and are solved in appendix C. The resulting Nusselt number is

$$N \sim [Ra^2A^2/(2k)^4]^{\frac{1}{3}}, \tag{4.42}$$

where A is the first derivative of the interior variable Ψ at $z = 0$. It was determined by Howard (1965) in his $C = 0$ analysis, and can be approximated very roughly by

$$A \approx \frac{2^{\frac{1}{2}}\pi}{\pi^2+a^2}\left[1+\frac{5}{2}\left(\frac{\pi^2+a^2}{9\pi^2+a^2}\right)^2\right]. \tag{4.43}$$

The function k is given by (4.41) and reduces to that found by Howard when $C = 0$. It should be pointed out that for real fluids the $R^{\frac{1}{3}}$ behaviour does not persist to extremely large R; for fixed σ, however large, the condition $\sigma \gg C(Ra^2)^{\frac{2}{3}}$ is not satisfied when R is large enough, and then $N \propto (R \ln R)^{\frac{1}{5}}$ as $R \to \infty$.

The dependence of N on a when $C \neq 0$ and $\sigma \ll C(Ra^2)^{\frac{2}{3}}$ is the same as when rigid boundary conditions are applied: N attains its maximum at $a = O(R^{\frac{1}{4}})$ when $\sigma R \gg 1$, and at $a = 2{\cdot}37$ when σR is small. When $\sigma \gg C(Ra^2)^{\frac{2}{3}}$ the dependence on a is the same as when $C = 0$ and once again N attains its maximum, proportional to $R^{\frac{1}{3}}$, when $a = O(1)$.

We note that in the lower boundary layer the vertical velocity amplitude varies linearly with z, whatever the value of σ. Thus the horizontal vorticity, which is proportional to $(D^2 - a^2)\,\Psi$, is small. This is not so in the upper boundary layer, however, where strong concentrations of vorticity occur, even though the stress at $z = 1$ is zero.

Finally, we observe that, as $R \rightarrow \infty$ with $\sigma R/C \ll 1$, (4.39) holds for free boundaries as well as rigid.

5. Discussion

The point of the present work has been to explore the possibility of describing nonlinear convection by treating its horizontal structure inaccurately while paying careful attention to its vertical structure. The way of doing this here has been to expand the horizontal variation of flow variables in terms of the planform functions of linear theory and then to truncate these expansions. The resulting equations differ from the mean-field equations by including a representation of the inertial terms. In this paper we have considered solutions of the single-mode equations (2.20)–(2.22) resulting from the severest truncation.

The case of mildly supercritical R treated in § 3 by finite amplitude expansions is principally of interest for comparison with known solutions of the full equations. We found that the single-mode solutions compare well with the corresponding solutions of the full equations for the case of hexagons. For rolls the agreement with the full solutions is good only for very large Prandtl number; this is just another manifestation of the failure of the mean-field equations ($C = 0$) to give a description of the dependence on Prandtl number of heat transport. Since $C = 0$ for rectangles as well, the single-mode representation fails in the same respect for these planforms. Thus, for moderate Rayleigh numbers accuracy is restricted to planforms for which $C \neq 0$, and otherwise to $\sigma \gg 1$. But in these cases reasonably good results can be obtained rather easily.

The asymptotic solutions of § 4 in the limit $R \rightarrow \infty$ display a dependence of $N - 1$ on R and σ, which we summarize in table 2 for various cases. The behaviour at very large σ, in the last column, results when the viscous terms dominate the inertial terms in the interior; in that case, for both rigid and free boundaries, the asymptotic forms are similar to those obtained from the single-mode mean-field approximation. The difference between the second and third columns arises from the different relative thicknesses of the vorticity and temperature boundary layers near the lower rigid boundary; there is no vorticity layer near a lower free boundary. (It should be recalled that the designations 'upper' and 'lower' refer to our arbitrary choice of solution with W, Θ and C positive; the designations are reversed when the sign of the product CW is changed.) When $\sigma \ll C$, as in the first and second columns, any vorticity boundary layer near the lower boundary is

	$\sigma \ll CR^{-1}$	$CR^{-1} \ll \sigma \ll C$	$C \ll \sigma \ll C(Ra^2)^{\frac{2}{3}}$	$\sigma \gg C(Ra^2)^{\frac{2}{3}}$
Rigid boundaries	$(\sigma R)^2$	$(\sigma R \ln \sigma R)^{\frac{1}{5}}$	$(R \ln R)^{\frac{1}{5}}$	$(R \ln R)^{\frac{1}{5}}$
Free boundaries	$(\sigma R)^2$	$(\sigma R \ln \sigma R)^{\frac{1}{5}}$	$(\sigma R \ln \sigma R)^{\frac{1}{5}}$	$R^{\frac{1}{3}}$

TABLE 2. Form of the asymptotic dependence of $N-1$ on R and σ for $R \to \infty$, for various ranges of σ. Although the rows are labelled according to the conditions on both boundaries, it is only the 'lower' boundary condition that determines the Nusselt number at leading order.

so thin compared with the thermal layer that the heat flux is independent of the boundary conditions on the stress. This is evident from table 1, where one can see that the integral k for a rigid boundary approaches the values for a free boundary as σ decreases. Further, when $\sigma \ll CR^{-1}$ thermal boundary layers cease to exist and the Nusselt number differs from unity by just a small amount; its value depends only on the solution in the interior, which is independent of the vorticity boundary layers.

The dependence of N upon a is not shown in table 2 since analytical functional forms are not available for all the cases. However, N appears to possess a single maximum N_m with respect to a in each case. This maximum occurs when $a = O(1)$ for free boundaries when $\sigma \gg C(Ra^2)^{\frac{2}{3}}$, and also for both rigid and free boundaries when $\sigma \ll CR^{-1}$. In all other cases it occurs when $a = O(R^{\frac{1}{4}})$.

The dependence of N on σ indicated in the first row of table 2 is of the kind hoped for in a qualitatively acceptable model. For $\sigma > 1$ experiments with rigid boundaries show little or no dependence of N on σ. However, as σ decreases through unity a slight decrease of N is found experimentally (Rossby 1969). The experiments unfortunately do not give us any guide as to the σ dependence for very small σ, but there is weak evidence from studies of stellar convection. Theoretical models of stars are computed on the assumption that for the very small Prandtl numbers of stellar material convective heat transfer does not depend on viscosity, and the models rationalize the observational data quite tolerably. The results presented here indicate that, for very small σ, N depends only on $R\sigma$, which accords very well with the astronomers' prejudices.

In this discussion of σ dependence we considered the wavenumber a to be fixed, while in fact we are at liberty to make it a function of R and σ. Unfortunately experimental data are too sparse for us to draw any firm conclusions concerning this function. However we recall that, at the highest R, N varies as R to a power which lies between 0·26 and 0·33 [see Rossby (1969) and Chu & Goldstein (1973) for summaries]. For rigid boundaries, if we choose $a = O(R^{\frac{1}{4}})$, we find that $N \propto R^{0 \cdot 30}(\ln R)^{0 \cdot 20}$, which certainly mimics the experimental data.

Unfortunately no experimental work has been done at the very large values of R at which our asymptotic analysis is valid, so we are able to compare our results only with other theoretical predictions such as the dimensional analysis that predicts $N \propto R^{\frac{1}{3}}$ as $R \to \infty$. This behaviour cannot be attained by the single-mode representation since a is at most $R^{\frac{1}{4}}$. But it appears to be a property of the multi-mode representation (Spiegel 1971), at least when the modes do not interact directly via the fluctuation advection terms, and such a behaviour has also been obtained by Chan (1971), who computed the maximum value of N permitted by the mean-field equations. Chan showed also that this maximum is an upper bound to the N permitted by the full Boussinesq equations with $\sigma = \infty$. However, it is not entirely clear that the $R^{\frac{1}{3}}$ law applies to real fluids. Kraichnan's (1961) analysis of convection at extremely high R predicts $N \propto [\sigma R(\ln R)^{-3}]^{\frac{1}{2}}$ when σ is small (but σR is large) and $N \propto \sigma^{-\frac{1}{4}}[R(\ln R)^{-3}]^{\frac{1}{2}}$ when σ is moderate. This steeper dependence on R arises from the increasing enhancement of the heat transport by boundary-layer turbulence as R increases. We do not know whether this phenomenon would be exhibited by our representation with many interacting modes, nor whether it even exists in real convection. But Kraichnan's argument is not implausible, and has not been contradicted by the Nusselt number maximization studies of Howard (1963) and Busse (1969).

Evidently, at very large R these single-mode equations are a very incomplete representation, and some of the properties seem unrealistic. For example, at high enough R, at least when a and σ are $O(1)$, the mean temperature field extends outside the range defined by the boundary conditions, as we noted above. But the purpose of both the mean-field equations and these modal equations is to describe the nonlinearities of convection in a tractable way. They may be viewed either as mathematical models for convection at high R, or alternatively as approximate representations of some aspects of convection at low R, where the scales of motion are fairly restricted.

To determine to what extent the single-mode equations may be useful at high R, more details of the experimental data must be compared with the theoretical predictions. Such details include the distributions of temperature and velocity, and the scales of motion which appear to dominate the heat transport. The asymptotic solutions presented here are valid only at values of R much greater than those realized in the laboratory, and detailed comparisons are deferred to a subsequent paper in which numerical solutions of the single-mode equations in the experimental range of R will be presented.

We are indebted to K. Grossman for providing numerical solutions of the lower boundary-layer equations and to L. N. Howard and J.-P. Zahn for some very valuable suggestions. We are grateful to L. Baker, W. S. Childress, J. B. Keller, J. D. Murray, E. L. Reiss and E. L. Schucking for their advice and interest.

The major part of this work was carried out in New York while D. O. G. and J. T. were at the Goddard Institute for Space Studies as NAS–NRC Resident Research Associates, and at the Courant Institute of Mathematical Sciences, New York University. Subsequently J. T. joined the Department of Mathematics, New York University. We are grateful to the National Science Foundation for financial support, most recently under NSF Grants GP-32336x and GA-43007, and to the National Aeronautics and Space Administration under NASA Contract NGL 06-003-057.

Appendix A. The outer asymptotic sequence

The discussion centring on (4.15) and (4.16) gives some idea of the first two terms of the appropriate outer sequence for the system (4.2)–(4.4). Further manipulations lead us to try an expansion of the form

$$\begin{aligned}\Psi(z,\epsilon) = {} & \Psi_0+\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}}[\Psi_{10}+\lambda^{-\frac{1}{3}}\Psi_{11}+\lambda^{-\frac{2}{3}}\Psi_{12}+\ldots] \\ & +\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\ln\lambda[\Psi_{l0}+\lambda^{-\frac{1}{3}}\Psi_{l1}+\ldots]+\ldots \\ & +(\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}})^2[\Psi_{20}+\ldots]+\epsilon\Psi_1+\ldots,\end{aligned} \tag{A 1}$$

with similar expansions for F and B. In (A 1) the various Ψ's on the right-hand side are independent of ϵ and λ.

The zeroth-order terms in this expansion lead to (4.15) with the choice $z_0 = 0$ and the behaviour of Ψ_0 near the boundaries expressed in (4.17) and (4.18).

We find next that the Ψ_{1n} ($n = 0, 1, 2, \ldots$) satisfy

$$\Psi_0^3(D^2-a^2)\Psi_{1n}-2z\Psi_{1n} = A_n\Psi_0, \tag{A 2}$$

where the A_n are arbitrary constants and in fact are the coefficients of the higher-order terms of an expansion of (4.16). Near $z = 0$ we see from (4.17) that (A 2) becomes, for $a = O(1)$,

$$z^2h^3(D^2-a^2)\Psi_{1n}-2\Psi_{1n} = A_nh, \tag{A 3}$$

where

$$h \simeq [3\ln z^{-1}-2\ln(3\ln z^{-1})]^{\frac{1}{3}}. \tag{A 4}$$

Near $z = 0$ we may use the approximation

$$D^2h = (zh)^{-2}, \tag{A 5}$$

so that (A 3) has the approximate solutions

$$\Psi_{1n} = -A_nh+B_nh^2, \tag{A 6}$$

where the B_n are arbitrary constants.

Similar considerations can be applied at the higher orders and the outer asymptotic sequence for Ψ in the neighbourhood of $z = 0$ is

$$\begin{aligned}\Psi = {} & zh+\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}}h^2(B_1+\lambda^{-\frac{1}{3}}B_2+\lambda^{-\frac{2}{3}}B_3+\ldots) \\ & -\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}}h(A_1+\lambda^{-\frac{1}{3}}A_2+\lambda^{-\frac{2}{3}}A_3+\ldots) \\ & +\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\ln(\lambda)\,h^2(B_{l1}+\lambda^{-\frac{1}{3}}B_{l2}+\ldots) \\ & -\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\ln(\lambda)\,h(A_{l1}+\lambda^{-\frac{1}{3}}A_{l2}+\ldots)-\epsilon\lambda^{\frac{5}{3}}A_1^2/2zh^2+\ldots \\ & +\epsilon(\tfrac{1}{2}C+\sigma/C)/zh+\ldots.\end{aligned} \tag{A 7}$$

Likewise, near $z = 1$, the asymptotic sequence can be written as

$$\begin{aligned}\Psi = {} & (1-z_0)^{\frac{1}{3}}(\tfrac{9}{2})^{\frac{1}{3}}(1-z)^{\frac{2}{3}}[1+K_2(1-z)^{\frac{2}{3}}+\tfrac{1}{3}(1-z_0)^{-1}(1-z)+\ldots] \\ & +\epsilon^{\frac{3}{5}}K_3(1-z)^{-\frac{1}{3}}+\ldots+\epsilon(4\sigma/15C)(1-z)^{-1}+\ldots.\end{aligned} \tag{A 8}$$

$$F = \frac{1}{zh}+\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}}\frac{1}{z^2h}[A_1-B_1h+\lambda^{-\frac{1}{3}}(A_2-B_2h)+\ldots]+\ldots, \tag{A 9}$$

and near $z = 1$

$$F = (1-z_0)^{-\frac{1}{3}}(\tfrac{2}{9})^{\frac{1}{3}}(1-z)^{-\frac{2}{3}}[1-K_2(1-z)^{\frac{2}{3}}+\ldots]+\ldots. \tag{A 10}$$

To facilitate the matching to the boundary-layer solution we express the interior solution in boundary-layer variables. Near $z = 0$, with $z = \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}\zeta$, we find

$$\begin{aligned}\Psi = {} & \epsilon^{\frac{1}{2}}\lambda^{\frac{3}{2}}B_1(1-\lambda^{-1}\ln\lambda - 2\lambda^{-1}\ln\zeta) \\ & +\epsilon^{\frac{1}{2}}\lambda^{\frac{7}{6}}[(B_2-A_1)-\lambda^{-1}\ln(\lambda)(B_2-\tfrac{1}{2}A_1)-\lambda^{-1}(2B_2-A_1)\ln\zeta] \\ & +\epsilon^{\frac{1}{2}}\lambda^{\frac{5}{6}}[(B_3-A_2)-\lambda^{-1}\ln(\lambda)(B_3-\tfrac{1}{2}A_2)-\lambda^{-1}(2B_3-A_2)\ln\zeta]+\dots \\ & +\epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}\ln(\lambda)B_{l1}(1-\lambda^{-1}\ln\lambda-2\lambda^{-1}\ln\zeta)+\dots \\ & +\epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}[(B_5-A_4+\zeta)-\lambda^{-1}\ln(\lambda)(B_5-\tfrac{1}{2}A_4-\tfrac{1}{2}\zeta) \\ & -\lambda^{-1}(2A_5-A_4-\zeta)\ln\zeta]+\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}\ln(\lambda)[(B_{l2}-A_{l1})+\dots]+\dots. \end{aligned} \qquad \text{(A 11)}$$

The leading term in this expansion is constant and $O(\epsilon^{\frac{1}{2}}\lambda^{\frac{3}{2}})$, for ζ fixed. We are unable to match such a term in the boundary layer, where for large ζ the leading term varies like ζ. We therefore must choose $B_1 = 0$, $B_2 = A_1$, $B_3 = A_2$, $B_4 = A_3$, $A_4 - B_5 = \text{constant} = \zeta_0$ (say) and $B_{l1} = \frac{1}{2}A_1$. This procedure leaves two arbitrary constants, A_1 and ζ_0, in the leading terms. For the higher-order terms a similar sequential cancellation is needed, but we consider here the matching of only the leading-order terms. We then find, near $z = 0$, that

$$\Psi = \epsilon^{\frac{1}{2}}\lambda^{\frac{1}{6}}(\zeta - A_1\ln\zeta - \zeta_0 + \dots) + O(\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{6}}) \qquad \text{(A 12)}$$

and

$$F = \epsilon^{-\frac{1}{2}}\lambda^{-\frac{1}{6}}\zeta^{-1}\left(1+\frac{A_1\ln\zeta}{\zeta}-\frac{\zeta_0}{\zeta}\right)+\dots, \qquad \text{(A 13)}$$

while near $z = 1$, with $\eta = \epsilon^{-\frac{3}{5}}(1-z)$, we find

$$\Psi = \epsilon^{\frac{2}{5}}[(\tfrac{9}{2})^{\frac{1}{3}}\eta^{\frac{2}{3}} + K_3\eta^{-\frac{1}{3}} - (4\sigma/15C)\eta^{-1}+\dots] + O(\epsilon^{\frac{4}{5}}) \qquad \text{(A 14)}$$

and

$$F = \epsilon^{-\frac{2}{5}}(\tfrac{2}{9})^{\frac{1}{3}}\eta^{-\frac{2}{3}}[1-(\tfrac{2}{9})^{\frac{1}{3}}K_3\eta^{-1}+(\tfrac{2}{9})^{\frac{1}{3}}(4\sigma/15C)\eta^{-\frac{5}{3}}+\dots]+O(1). \qquad \text{(A 15)}$$

These expressions provide the matching conditions to be applied to the boundary-layer solutions.

Appendix B. Solution of (4.15) for large a

The leading term in the outer asymptotic sequence satisfies (4.15) with $z_0 = 0$,

$$\Psi_0^2(D^2-a^2)\Psi_0 = -z,$$

where $D = d/dz$. We need to know how Ψ_0 behaves near the boundaries ($z = 0, 1$), and in the text we have assumed that Ψ_0 vanishes on both boundaries. These conditions lead to a successful matching. Here we simply wish to verify the behaviour indicated in (4.17) and (4.28) and shall do so for $a \gg 1$.

For large a the asymptotic solution of (4.15) away from the boundaries ($z = 0, 1$) is

$$\Psi_0 = \left(\frac{z}{a^2}\right)^{\frac{1}{3}}\left[1-\frac{2}{27a^2}z^{-2}+\dots\right]. \qquad \text{(B 1)}$$

To discuss the behaviour of the solutions near $z = 1$, we introduce new variables:

$$1-z = a^{-1}r, \quad \Psi_0 = a^{-\frac{2}{3}}\Phi. \qquad \text{(B 2)}$$

Now (4.15) becomes

$$\Phi^2(d^2/dr^2-1)\Phi + 1 = a^{-1}r. \qquad \text{(B 3)}$$

If we introduce the expansion

$$\Phi = \Phi_0 + a^{-1}\Phi_1 + \dots \tag{B 4}$$

into (B 3) we readily obtain the equation for Φ_0. From (B 1) we see that $\Phi_0 \to 1$ as $r \to \infty$ and that the appropriate solution for the Φ_0 equation is

$$r = 2\left[\frac{1}{3^{\frac{1}{2}}}\tanh^{-1}\left(\frac{3\Phi_0}{\Phi_0+2}\right)^{\frac{1}{2}} - \tanh^{-1}\left(\frac{\Phi_0}{\Phi_0+2}\right)^{\frac{1}{2}}\right]. \tag{B 5}$$

Near $r = 0$, we expect $\Phi_0 \approx 0$ and we can invert (B 5) to yield

$$\Phi_0 = 2(\tfrac{3}{4}r)^{\frac{2}{3}}\,[1 - \tfrac{3}{5}(\tfrac{3}{4}r)^{\frac{2}{3}} + \dots]. \tag{B 6}$$

On expressing this in terms of z we find that, near $z = 1$,

$$\Psi_0 = (\tfrac{9}{2})^{\frac{1}{3}}(1-z)^{\frac{2}{3}}\,[1 - \tfrac{3}{10}(\tfrac{9}{2})^{\frac{1}{3}}a^{\frac{2}{3}}(1-z)^{\frac{2}{3}} + \dots], \tag{B 7}$$

which agrees with (4.18) and establishes the value of K_2 for $a \gg 1$.

Near $z = 0$ we scale as follows:

$$z = a^{-1}s, \quad \Psi_0 = a^{-1}X. \tag{B 8}$$

With these scaled variables we can rewrite (B 1) as

$$X = s^{\frac{1}{3}}(1 - 2/27s^3 + \dots) \quad \text{for} \quad s \to \infty. \tag{B 9}$$

The 'boundary layer' equation is

$$X^2(d^2/ds^2 - 1)\,X + s = 0, \tag{B 10}$$

which is unfortunately not much simpler than (4.15). However, we can verify that as $s \to \infty$ it admits solutions which match to (B 9). Moreover, as $s \to 0$ it has a solution which when expressed in z goes to zero like

$$\Psi_0 = z(3\ln z^{-1} - 2\ln\ln z^{-1} - 3\ln a + \dots)^{\frac{1}{3}}. \tag{B 11}$$

This agrees with (4.17) if we set $K_1 = a^{-1}$. Thus we see that the constants K_1 and K_2 are determined from the interior equation when the conditions $\Psi_0(0) = \Psi_0(1) = 0$ are applied. To obtain better values of K_1 and K_2 for $a = O(1)$ we would need to solve (3.15) numerically; however, we have no real need of such precision.

Appendix C. The free boundary layer

As indicated in the text, the solution of (4.27) with the conditions (4.40) is $\psi \simeq \zeta$. Thus (4.32) becomes

$$f'' - 2C\zeta f' - (C + \zeta^2)f = -\zeta \tag{C 1}$$

and we require $f(0) = 0$ and $f \sim \zeta^{-1}$ as $\zeta \to \infty$. This equation is very similar to the one which arises in the case $C = 0$ treated by Howard (1965) and it is not hard to extend his treatment and find the solution in integral form. This is

$$f = \tfrac{1}{2}C^{\frac{1}{2}}\zeta\, e^{\frac{1}{2}C\zeta^2}\int_1^{\mu}(\mu^2 - t^2)^{-\frac{1}{4}}\, e^{-\frac{1}{2}C\zeta^2 t}\, dt, \tag{C 2}$$

where

$$\mu^2 = (C^2+1)/C^2. \tag{C 3}$$

The evaluation of N proceeds as before and we obtain (4.38), but this time

$$k = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\left\{[C(\mu-1)]^{-\frac{1}{2}} + \frac{1}{2C}\int_1^\mu \frac{C^{\frac{1}{2}} - (\mu^2-t^2)^{-\frac{1}{2}}}{(t-1)^{\frac{3}{2}}}\,dt\right\}. \tag{C 4}$$

With some standard manipulations this can be cast into lengthy but known integrals, and we are led to (4.41).

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